Solutions of Assignment 6

Q1. (a) When $\alpha = 1$, we can directly verify the convexity of $|x|$ by considering for any $\lambda \in [0,1]$ and for all $x, y \in \mathbb{R}$, then

$$|\lambda x + (1-\lambda)y| \leq \lambda|x| + (1-\lambda)|y| \qquad \text{(triangle inequality)}$$

When $\alpha > 1$, $f(x) = x^\alpha$ is twice-differentiable and $f''(x) = \alpha(\alpha-1)x^{\alpha-2} \geq 0$. Then, the result follows from second-order condition.

(b) We compute

$$\mathrm{Hess}(f) = \begin{pmatrix} \dfrac{2}{y} & -\dfrac{2x}{y^2} \\ -\dfrac{2x}{y^2} & \dfrac{2x^2}{y^3} \end{pmatrix} = \frac{2}{y^3}\begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}$$

Note that $\dfrac{2}{y^3}\left|y^2\right| = \dfrac{2}{y} \geq 0$ and $\left(\dfrac{2}{y^3}\right)^2 \begin{vmatrix} y^2 & -xy \\ -xy & x^2 \end{vmatrix} \geq 0.$

So $\mathrm{Hess}(f)$ is positive semi-definite, for all $(x,y) \in \mathbb{R} \times (0,\infty)$, $f(x,y)$ is convex on $\mathbb{R} \times (0,+\infty)$.

(c) Note that

$$\nabla f(x) = Px + q \qquad \text{and} \qquad \mathrm{Hess}(f) = P$$

which is symmetric and positive semi-definite, and use second-order condition, $f(x)$ is convex.

Q2. (a) We separate into following cases:

- **Case 1:** $x \notin X$

  Then, we have $I(x) = +\infty$, so by convention, $\partial I(x) = \emptyset$ because no finite subgradient exists.

- **Case 2:** $x \in X$

  Then $I(x) = 0$, so for any $y \in \mathbb{R}^N$, the subgradient inequality becomes:

$$I(y) \geq w^T(y-x)$$

If $y \in X$, then $I(y) = 0$ yields $0 \geq g^T(y-x) \iff g^T(y-x) \leq 0$.

If $y \notin X$, then $I(y) = +\infty$ and the inequality holds automatically.

So, the condition $w^T(y-x) \leq 0$ holds for all $y \in X$, and this defines the **normal cone** to X at x, denoted as follows:

$$N_X(x) = \left\{ w \in \mathbb{R}^N : w^T(y-x) \leq 0,\ \forall y \in X \right\}$$

Thus, the subdifferential of the indicator function $I(x)$ is

$$\partial f(x) = \begin{cases} N_X(x), & \text{if } x \in X, \\ \emptyset, & \text{if } x \notin X. \end{cases}$$

where $N_X(x)$ is the normal cone to X at x.

(b) For $x \in (-2,1), (-1,1)$ and $(1,2)$, f is differentiable, hence $\partial f(x) = \{\nabla f(x)\}$.
For $x \in (-\infty,-2) \cup (2,\infty)$, $f(x) = \infty$, hence $\partial f(x) = \emptyset$.
For $x = 1$, we show that $\partial f(x) = [0,1]$.
Let $w \in \partial f(1)$, then

$$f(y) \geq w(x-1) \quad \forall y \in \mathbb{R}$$

If $y \in [1,2]$, then $x - 1 \geq w(x-1)$, that is $1 \geq x$.
If $y \in [-1,1]$, then $0 \geq w(x-1)$, so $w(1-x) \geq 0$ and $w \geq 0$.
It is easy to check that for $w \in [0,1]$, w satisfies

$$f(y) \geq f(1) + w(x-1), \;\; \forall y$$

Hence $\partial f(1) = [0,1]$.
Thus, we have

$$\partial f(x) = \begin{cases} \emptyset & x \in (-\infty,-2) \cup (2,\infty) \\ (-\infty,-1] & x = -2 \\ \{-1\} & x \in (-2,-1) \\ [-1,0] & x = -1 \\ \{0\} & x \in (-1,1) \\ [0,1] & x = 1 \\ \{1\} & x \in (1,2) \\ [1,\infty) & x = 2 \end{cases}.$$

Q3. (1) Consider $f(x) = \frac{1}{x}$, $\operatorname{dom} f = (0,+\infty)$, $K = (0,1]$.
Since $f'(x) = -\frac{1}{x^2}$ is unbounded on K, f is not Lipschitz continuous.

(2) Consider $f(x) = x^2$, $\operatorname{dom} f = K = \mathbb{R}$. Since $f'(x) = 2x$ is unbounded on $K = \mathbb{R}$, the function is not Lipschitz continuous on K.

(3) Consider $f(x) = -\sqrt{x}$, $\operatorname{dom} f = [0,+\infty)$, $K = [0,1]$. Since

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} -\frac{1}{\sqrt{x}}$$

does not exist, f is not Lipschitz continuous on K.